Abstract

The aim of this thesis is to understand the Leray-Serre spectral sequence of a bration and use it to compute cohomology of some interesting manifolds. To do that, it is important to understand spectral sequences rst. So the rst part of my thesis is devoted to an introduction to spectral sequences and then some background in topology to understand the Leray-Serre spectral sequence of a bration. A cohomology spectral sequence is a collection of di erential bigraded Rmodules fE ; r ;drg; r = 1;2; : : :, where the di erentials are all of bidegree (r;1r) such that for all r, the E ; r+1-term is given as the cohomology of the E ; r -term. Pictorially one can imagine this as a three dimensional lattice with each lattice point an R-module, the di erentials as arrows between them and each page is obtained by taking the cohomology of the previous page. One can observe that, the knowledge of E ; r and dr determines E ; r+1, but not dr+1. So if a di erential is not known then one needs some other method to proceed. The rst property that a spectral sequence admits is that it can be represented as an in nite tower of submodules of the E2-term and conversely. Thus one can de ne the limit term of this sequence which we call the E1-term. Now the ultimate goal is to compute this E1-term. It is interesting to note that if a spectral sequence 'collapses' at, say N, then the computation of E1-term becomes easy as the sequence becomes constant after (N 1)th page. Once we know what a spectral sequence is, the natural question that one can ask is how can we construct one? In this direction, there are two general algebraic settings in which spectral sequences arise naturally. First is a ltered di erential graded module and second is an exact couple. In the rst case, each ltered di erential graded module A determines a spectral sequence with di erential of bidegree (r;1r) and if the ltration is bounded then the spectral sequence converges to H(A;d) (the homology of A with respect to d). This result rst appeared in the work of Koszul [3] and Cartan [1]. There are also weaker conditions which ensure the convergence and uniqueness of the target. Thus if the ltration is exhaustive and weakly convergent, the same result will still hold true. Second case is that of an exact couple. This idea was introduced by Massey [5]. An exact couple also determines a spectral sequence of cohomological type. It is interesting to observe that one can also associate a tower of submodules of E and an E1-term to an exact couple, just as we can do for any spectral sequence. The next question one could ask is that if the two approaches are related in any way? and if yes then how do the two spectral sequences compare? The answer to the above question is yes and it is not very di cult to see that a ltered di erential graded module gives rise to an exact couple. And in fact, the two spectral sequences, one associated to the ltered di erential graded module and the other associated to the exact couple derived from the ltered di erential graded module, turn out to be same. There is another algebraic object namely double complex which gives rise to two spectral sequences, which in turn help in the calculation of the homology of the total complex associated to a double complex. Double complexes o er an example of the ltered di erential graded module construction of a spectral sequence. Finally, with enough background on spectral sequences, one can talk about brations and the spectral sequence associated to them. A map satisfying the homotopy lifting property with respect to all spaces is called a Hurewicz bration (or just a bration), while a map with the homotopy lifting property with respect to all ncells is called a Serre bration. It was Leray [4] who solved the problem of relating the cohomology rings of spaces making up a ber space by developing a powerful computational gadget called a spectral sequence. This spectral sequence has many applications such as computation of cohomology of various Lie groups, homogeneous spaces and loop spaces.