Abstract

The main aim of my thesis is to review the major developments in the area of integral and modular dimension subgroups and study some of their applications. One of the fundamental objects of study in group theory is the lower central series. Magnus [Mag35] was one of the first to investigate the lower central series of free groups. To recall his approach, let F be a free group ring with basis $X = \{x_i\}_{i \in I}$. Let $A = Z[[X_i]]$ be the ring of formal power series and U(A) the group of units of A. Clearly, $1 + X_i$ is an invertible element with the inverse as $1 - X_i + X_i^2 - \ldots$. The map $x_i \mapsto 1+X_i$ extends to a homomorphism $\theta : F \to U(A)$. It can be shown that θ is actually a monomorphism [MKS76, Chapter 5]. For $a \in A$, let a_n be the homogeneous components of degree n so that $a = a_0 + a_1 + \ldots + a_n + \ldots$. Magnus defined dimension subgroups, $D_n(F)$, $n \geq 1$, as follows $D_n(F) := \{f \in F \mid \theta(f) = 1 + \theta(f_n) + \theta(f_{n+1}) + \ldots\}$. (1) These subgroups are normal subgroup with the property that $(F,D_n(F)) \subseteq D_{n+1}(F)$ for all $n \geq 1$ where for M,N subgroups of the group, G, we define (M,N) to be the subgroup generated by the commutators $(m, n) = m^{-1}n^{-1}mn$ for $m \in M$ and $n \in N$, i.e., $(M,N) = \langle (m, n) = m^{-1}n^{-1}mn \mid m \in M \text{ and } n \in N \rangle$. (2) Let f be the augmentation ideal of Z[F]. Define $D_n(F) = G \cap (1+f^n)$. For free groups, it is easy to see that $\gamma_n(F) \subseteq D_n(F) \subseteq D_n(F)$ for all $n \geq 1$. The homomorphism θ can be extended to a monomorphism : $Z[F] \to U(A)$. Under this map, $\alpha \in f^n$ maps to an element where $\theta(\alpha)_i = 0$ for all $i \leq n - 1$. From thework of Grün [Gru36], Magnus [Mag37] and Witt [Wit37] it follows that the above inclusions are actually equalities i.e., $\gamma_n(F) = D_n(F) = D_n(F)$ for all $n \geq 1$. (3) The above result gives a close relation between the lower central series and the dimension series. It was only natural to conjecture that, for any group G, the lower central series and the dimension series coincide. It was in 1972 that E. Rips [Rip72] settled this conjecture by giving a counter-example. In the first chapter, we study integral dimension subgroups. We see that the first three terms of the integral dimension series and the lower central series of an arbitrary group coincide; however, beyond that the equality does not hold in general. We study the structure of the fourth [Tah77] and the fifth dimension subgroups [Tah81] in some detail. We also study some of the counter-examples given by Gupta [Gup90]. In the second part of this chapter we focus on dimension subgroups over fields. In the second chapter, we study the Lie dimension subgroups, $D_{(n)}[G]$ and $D_{[n]}(G)$ for $n \geq 1$. We see that $\gamma_n(G) \subseteq D_{[n]}(G) \subseteq D_{(n)}(G) \subseteq D_n(G)$. We explore the Lie dimension subgroups in some detail to realize that more definitive results are known about them. We also discuss the identification of Lie dimension subgroups over fields as given by Passi and Sehgal [PS75]. In the last chapter, we study powerful p-groups which were introduced by Lubotzky and Mann [LM87]. These can be thought of as generalization of Abelian groups. Shalev [Sha90] introduced a double-indexed series, $\{D_{m,k}\}$, which we study in some detail. We focus on some of its properties and see how these are related to dimension subgroups [SS91]. We discuss how powerful and potent p-groups help us understand the power structure of p groups [Wil03]. Extensive work has been done in this area by A. Shalev [Sha90] [Sha91], C. Scoppola [Sco91] and others.